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Nearly Ferromagnetic Superconductors

Qi Li and D. Belitz
Department of Physics and Materials Science Institute, University of Oregon, Eugene, OR 97403, USA

T.R. Kirkpatrick
Institute for Physical Science and Technology, and Department of Physics, University of Maryland, College Park, MD 20742, USA

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The electromagnetic properties of superconductors near a ferromagnetic instability are investigated by means of a generalized Ginzburg-Landau theory. It is found that the magnetic flux expulsion capability of the superconductor gets stronger, in a well-defined sense, as the normal-state magnetic susceptibility increases. The temperature dependencies of the London penetration depth, the critical fields, and the critical current are all strongly affected by ferromagnetic fluctuations. In particular, for the critical current we find a temperature exponent $\alpha \approx 2$ over an appreciable temperature range. The extent to which proximity to magnetic criticality may be a viable explanation for recent observations in MgCNi microfibers, which find $\alpha \approx 2$, is discussed.

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I. INTRODUCTION

The coexistence of ferromagnetism and superconductivity has received substantial attention over the years. Around 1980, such states were predicted[1,2,3] and observed,[4,5] and the topic later received renewed interest in the context of experimental observations in rare earth borocarbides.[6] More recently, interest in this subject has been revived by the observation of coexisting superconductivity and ferromagnetism in UGe_2[7,8] and URhGe,[9] where both types of order are believed to be due to electrons in the same band. Recent theoretical attention has centered on the structure of the phase diagram,[10] on the existence of spontaneous flux lattices,[6,11,12] and on the question of spin-triplet versus spin-singlet superconductivity.[13]

In contrast, less is known about the properties of superconductors on the paramagnetic side of, but close to, a ferromagnetic instability. We will refer to "paramagnetic superconductors" to describe systems in this regime, although the superconductivity of course leads to the usual strong diamagnetic effects. Such paramagnetic superconductors include systems below the superconducting transition temperature, but above the temperature below which coexistence of superconductivity and ferromagnetism occurs, as well as systems that never develop ferromagnetism, but are close to a ferromagnetic instability in some direction in parameter space other than temperature. An example of the latter is believed to be the non-oxide perovskite $MgCNi_3$, which superconducts below a critical temperature $T_c \approx 8\,K$.[14] There is no evidence for a ferromagnetic phase in this material, but it has been suggested that a ferromagnetic ground state can be reached upon a relatively small amount of hole doping.[15] This system may thus be close to a ferromagnetic instability everywhere in its superconducting phase.

A recent study of $MgCNi_3$ microfibers, with $T_c = 7.8\,K$, has revealed an anomalous temperature dependence of the critical current density j_c.[16] The critical current density vanishes at T_c according to a power law $j_c \propto |T - T_c|^\alpha$, with $\alpha = 2$ between about 1% and 10% away from the critical point, and no crossover to the usual Ginzburg-Landau behavior, which predicts $\alpha = 3/2$. The authors of Ref. 16 have ruled out morphological effects as an explanation, which raises the question whether proximity to a ferromagnetic state may be responsible. Indeed, since ferromagnetic fluctuations are expected to weaken (singlet) superconductivity, this is a plausible suggestion for the origin of the weaker-than-expected temperature dependence of j_c.

The probable proximity to ferromagnetism has led to a debate about the nature and symmetry of the pairing in $MgCNi_3$.[17] This point has not been settled; some experimental evidence points to conventional s-wave pairing; other, to a superconducting order parameter with nodes. The nature of the pairing in the other materials mentioned above has not been unambiguously determined either. In this paper we will focus on the behavior close to T_c, which is qualitatively independent of the symmetry of the order parameter and thus expected to be the same for all nearly ferromagnetic superconductors. We use a generalized Ginzburg-Landau theory to theoretically investigate the electrodynamic properties of a superconductor as a ferromagnetic instability is approached. We treat the superconductivity in the usual mean-field approximation, but the magnetic critical behavior exactly in a scaling sense. Somewhat counter-intuitively, strong magnetic fluctuations make, in a well-defined sense, the superconductivity more robust in certain respects. In particular, the penetration depth becomes anomalously short. The thermodynamic critical field, on the other hand, becomes weaker, as one might intuitively expect. The temperature dependencies of the critical field H_c and the penetration depth λ depend on the magnetic critical exponents δ and γ, respectively. For the critical current $j_c \propto H_c/\lambda$, this results in an exponent α between 1.5 (the

Ginzburg-Landau result) and 2.16 in various temperature regimes. We will discuss both the existing experimental observations, and predictions for new experiments, in the light of these results.

This paper is organized as follows. In Sec. II we give elementary phenomenological arguments for the dependence of the thermodynamic critical field, the penetration depth, and the critical current density, on a constant normal-state magnetic permeability $\mu_{\rm n}$. We then generalize these results to the magnetically critical case, where one needs to distinguish between $\mu_{\rm n}$ and the spin susceptibility $\mu_{\rm s}$ in a superconduting state, and both $\mu_{\rm n}$ and $\mu_{\rm s}$ become nonanalytic functions of various control parameters. In Sec. III we derive these results from a generalized Ginzburg-Landau theory, and in Sec. IV we give a discussion of our results.

II. PHENOMENOLOGICAL ARGUMENTS

A. Paramagnetic systems

We are interested in the electromagnetic properties of superconductors with ferromagnetic fluctuations. We denote the normal-state spin susceptibility, which describes the response of the spin degrees of freedom to an external magnetic field in the absence of superconductivity, by $\chi_{\rm n}$, and the corresponding spin permeability by $\mu_{\rm n} = 1 + 4\pi\chi_{\rm n}$. This is in contrast to the spin permeability $\mu_{\rm s} = 1 + 4\pi\chi_{\rm s}$, which includes the effects of the superconductivity on the spin response, and the magnetic permeability $\mu = 1+4\pi\chi$, which describes the response of the total magnetization, including the diamagnetic part. It is instructive to first recall the dependence of superconducting properties on a constant $\mu_{\rm n} \neq 1$, neglecting the distinction between $\mu_{\rm n}$ and $\mu_{\rm s}$.[18,19] This can be done by means of elementary arguments.

1. *Thermodynamic critical field*

Consider the free energy density f of a system in a magnetic field. It obeys

$$df = df(H=0) + \frac{1}{4\pi}\, H\, dB, \tag{2.1}$$

where H is the thermodynamic magnetic field, and B is the magnetic induction. For the sake of simplicity, we ignore the vector nature of various quantities in our free energy considerations. For fixed B, f is the appropriate thermodynamic potential whose minimum determines the equilibrium state. However, in an experiment H is fixed, since $(c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{j}_{\rm ext}$ is the external current density, and only the latter is experimentally controlled. One therefore must perform a Legendre transform to a thermodynamic potential $g = f - BH/4\pi$,[20,21] which obeys

$$dg = df(H=0) - \frac{1}{4\pi}\, B\, dH. \tag{2.2}$$

In a paramagnetic phase, including paramagnetic superconductors, the relation between B and H is

$$B = H + 4\pi M = (1 + 4\pi\chi)H = \mu H, \tag{2.3}$$

with M the magnetization, $\chi(T,H)$ the magnetic susceptibility, and $\mu = 1 + 4\pi\chi$ the magnetic permeability. Integration of Eq. (2.2) yields

$$g(T,H) = f(T,H=0) - \frac{1}{4\pi}\int_0^H dh\, \left[1 + 4\pi\chi(T,h)\right] h. \tag{2.4}$$

This is generally valid. In a superconducting Meissner state, $B = 0$, and hence $\chi = -1/4\pi$ (ideal diamagnetism), and $f(T,H=0) = f_0 + t|\psi|^2/2 + u|\psi|^4/4$, with f_0 the free energy density of the normal state, ψ the superconducting order parameter, $t \propto (T - T_{\rm c})/T_{\rm c}$ the dimensionless distance from the superconducting critical point, and u a parameter. In a normal metal far from a magnetic instability, and ignoring normal-state diamagnetic effects, $\chi(T,H) \approx$ const. $\equiv \chi_{\rm n}$, or $\mu_{\rm n} = 1 + 4\pi\chi_{\rm n} =$ const., and $f(T,H=0) = f_0$. In a normal metal close to a ferromagnetic critical point, χ is a complicated function of T and H.

Now consider a superconductor with $\mu_{\rm n} =$ const. According to Eq. (2.4), the magnetic energy density gained by the system allowing magnetic flux to penetrate, i.e., the free energy density difference between the Meissner state with $B = 0$ and the normal state with $B = \mu_{\rm n}H$, is $E_{\rm m}/V = \mu_{\rm n}H^2/8\pi$. By contrast, the condensation energy density gained by the system becoming a superconductor is $E_{\rm cond}/V = t^2/4u$. The thermodynamic critical field, which is defined by these two energies being equal, is thus

$$H_{\rm c} = \sqrt{2\pi/u}\;|t|/\sqrt{\mu_{\rm n}} = H_{\rm c}^0/\sqrt{\mu_{\rm n}}, \tag{2.5}$$

with $H_{\rm c}^0 = \sqrt{2\pi/u}\,|t|$ the critical field for a system with $\mu_{\rm n} = 1$. An increase in $\mu_{\rm n}$ thus decreases the critical field, as one might expect since the externally applied field is amplified inside the material.

2. *London penetration depth*

The dependence of the London penetration depth λ on $\mu_{\rm n}$ is intuitively less obvious. Consider a large superconducting sample, with linear dimension L, surrounded by vacuum and subject to a homogeneous external magnetic field $\boldsymbol{H} = (0,0,H)$ in z-direction. Along the left edge of the sample, the magnetic induction will be of the form $\boldsymbol{B}(\boldsymbol{x}) = (0,0,B(x))$ with $B(x) = B_0\, e^{-x/\lambda}$ $(x > 0)$. To determine B_0, imagine a thin (thickness d) layer of normal conducting material around the superconductor. Except for the superconductivity, the normal

layer should have the same properties as the superconductor, in particular, a magnetic permeability $\mu_{\rm n}$. Then we have $B = H$ in vacuum, and $B = \mu_{\rm n} H$ inside the normal layer, see Fig. 1(a). Now let $d \to 0$. Then we

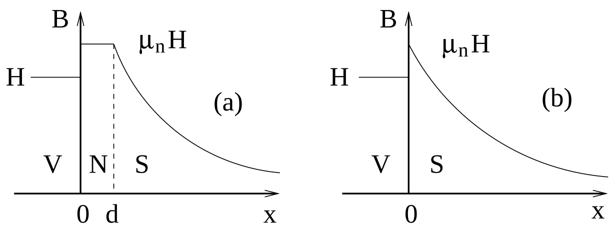

FIG. 1: Magnetic induction schematically as a function of position at a vacuum (V) - normal metal (N) - superconductor interface (S) (a), and at a vacuum - superconductor interface (b).

have (Fig. 1(b))

$$B(x) = \begin{cases} H & \text{for} \quad x < 0 \\ \mu_{\rm n} H\, e^{-x/\lambda} & \text{for} \quad x \geq 0 \end{cases} . \tag{2.6}$$

Now consider the current density associated with $B(x)$. From Ampere's law we have

$$\boldsymbol{j}(\boldsymbol{x}) = \frac{c}{4\pi} \boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{x}) = (0, j(x), 0), \tag{2.7a}$$

with c the speed of light and

$$j(x) = \frac{c}{4\pi\lambda}\, B(x). \tag{2.7b}$$

This is the *total* current density. It has three contributions, namely, the supercurrent density $\boldsymbol{j}_{\rm sc}$, the spin or magnetization current $\boldsymbol{j}_{\rm spin} = c\boldsymbol{\nabla} \times \boldsymbol{M}$, with $\boldsymbol{M}$ the spin contribution to the total magnetization, and the external current density $\boldsymbol{j}_{\rm ext} = c\boldsymbol{\nabla}\times\boldsymbol{H}/4\pi$. The latter vanishes in the case we are considering. In a normal metal, the spin current is the only contribution if we ignore normal-state diamagnetic effects. The spin or normal-state susceptibility $\chi_{\rm n}$ is defined as the response of $\boldsymbol{M}$ to the total magnetic induction $\boldsymbol{B}$ minus the contribution to $\boldsymbol{B}$ of $\boldsymbol{M}$ itself,

$$\boldsymbol{M} = \chi_{\rm n}(\boldsymbol{B} - 4\pi\boldsymbol{M}) = (\chi_{\rm n}/\mu_{\rm n})\, \boldsymbol{B}. \tag{2.8}$$

For the supercurrent density $\boldsymbol{j}_{\rm sc} = \boldsymbol{j} - \boldsymbol{j}_{\rm spin}$ this implies

$$\boldsymbol{j}_{\rm sc} = \frac{c}{4\pi\mu_{\rm n}}\, \boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{x}) = (0, j_{\rm sc}(x), 0), \tag{2.9a}$$

with

$$j_{\rm sc}(x) = \frac{c}{4\pi\mu_{\rm n}\lambda}\, B(x). \tag{2.9b}$$

Now consider one surface (area L^2) of the sample. Neglecting corner effects, and for $\lambda \ll L$, the total magnetic flux Φ through that surface is

$$\Phi = \int_0^L dy \int_0^L dx\, B(x) \approx L \int_0^\infty dx\, B(x) = L\lambda\mu_{\rm n} H. \tag{2.10}$$

On the other hand, the total supercurrent flowing near that surface is, from Eq. (2.9b),

$$I_{\rm sc} = \int d\boldsymbol{x}\, j_{\rm sc} \approx L^2 \int_0^\infty dx\, j_{\rm sc}(x) = \frac{c}{4\pi\mu_{\rm n}}\, \frac{L}{\lambda}\, \Phi. \tag{2.11}$$

We thus can write the flux

$$\Phi = \frac{4\pi\mu_{\rm n}}{c}\, \frac{\lambda}{L}\, I_{\rm sc} = \frac{4\pi\mu_{\rm n}}{c}\, \frac{\lambda}{L}\, Nqv, \tag{2.12}$$

where N is the number of supercurrent carrying particles, q is their charge, and v is their velocity. If m is their mass, then $E_{\rm kin} = Nmv^2/2$ is the total kinetic energy of the supercurrent. The flux can thus be written

$$\Phi = \frac{4\pi\mu_{\rm n}}{c}\, \frac{\lambda}{L}\, q\sqrt{2/m}\sqrt{N}\sqrt{E_{\rm kin}}. \tag{2.13}$$

Now we make two observations. First, $N \approx L^2\lambda n$, with $n = |\psi|^2$ the particle number density. Second, at the critical field strength the kinetic energy of the supercurrent must equal the condensation energy in the region where the current is flowing, which is (see Sec. II A) $E_{\rm cond} = L^2\lambda t^2/u$. With λ_0 the London penetration depth for $\mu_{\rm n} = 1$,

$$\lambda_0 = \sqrt{mc^2/4\pi q^2|\psi|^2} \propto 1/|t|^{1/2}, \tag{2.14}$$

this allows to write the flux at the critical field

$$\Phi_{\rm c} = \lambda L H_{\rm c}^0 \mu_{\rm n}\lambda/\lambda_0 = L\lambda\, \mu_{\rm n} H_{\rm c}, \tag{2.15}$$

where the first equality follows from Eq. (2.13), and the second one from Eq. (2.10). We thus obtain

$$H_{\rm c}/\lambda = H_{\rm c}^0/\lambda_0\ , \tag{2.16a}$$

or[18,19]

$$\lambda = \lambda_0/\sqrt{\mu_{\rm n}}\ . \tag{2.16b}$$

The penetration depth thus *decreases* with increasing $\mu_{\rm n}$, as does the critical field. This is somewhat counterintuitive, as it implies that the superconductivity becomes in some sense more robust. It also implies that a large normal-state magnetic permeability will make the superconductor necessarily of type I.[2] We will come back to this observation.

Notice that the above derivation relies only on very general energetic considerations and on Ampere's law. Also notice that it uses an identity at the critical field strength, where the superconductivity vanishes. This is fine for $H_{\rm c}$, but the penetration depth is a property of the superconducting state, and hence the use of $\mu_{\rm n}$ is not quite appropriate for this quantity, except in the limit $\lambda \to \infty$. More generally, λ depends on $\mu_{\rm s}$, which in turn depends on the superconducting properties. This makes no difference deep inside the paramagnetic superconducting phase, and Eq. (2.16b) is valid there. However, as we will see it makes a crucial difference close to a ferromagnetic instability.

3. Critical current

In order to discuss the critical current, we assume a thin-wire geometry with wire radius R.[20,21] The supercurrent density, which is the total current density minus the spin current density, can be written as a generalization of Eq. (2.9a),

$$\begin{aligned} j_{\rm sc}(\boldsymbol{x}) &= \frac{c}{4\pi}\left[\boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{x}) - 4\pi\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{x})\right] \\ &= \frac{c}{4\pi\mu(\boldsymbol{x})}\,\boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{x}), \end{aligned} \tag{2.17}$$

where we have used Eq. (2.3) and $\mu(\boldsymbol{x})$ is the local magnetic susceptibility. Now integrate over the cross section of the wire. Assuming a homogeneous current density within a distance λ from the surface, and using Gauss's theorem on the right-hand side, we have

$$2\pi R\lambda j_{\rm sc} = \frac{c}{4\pi\mu_{\rm n}}\oint d\boldsymbol{\ell}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{x})) = \frac{c}{2}\,RH$$

where we have used Eq. (2.6). The critical current density $j_{\rm c}$ is the one that produces the thermodynamic critical field $H_{\rm c}$, which yields the familiar London theory result[20]

$$j_{\rm c} = cH_{\rm c}/4\pi\lambda. \tag{2.18}$$

This result is plausible: Dimensionally, $j_{\rm c}$ must be a magnetic field divided by a length. The relevant length scale is the thickness of the area that supports diamagnetic currents, which is λ. The relevant field scale should be the field that corresponds to the condensation energy, which is $H_{\rm c}$. To the extent that $\mu_{\rm s} \approx \mu_{\rm n} = \text{const.}$, as we have assumed in Sec. II A 2, Eq. (2.16a) implies that $j_{\rm c}$ is independent of $\mu_{\rm n}$,

$$j_{\rm c} = j_{\rm c}^{\,0}. \tag{2.19}$$

As we will see below, this result changes drastically in the vicinity of a ferromagnetic instability.

B. Systems at a ferromagnetic instability

In the vicinity of a ferromagnetic instability of the normal metal, the normal state magnetic susceptibility $\chi_{\rm n}$, and hence the permeability $\mu_{\rm n}$, become large and diverge as the phase transition is approached. At a ferromagnetic critical point, the region of linear response shrinks to zero, and $\chi_{\rm n}$ and $\mu_{\rm n}$ become strongly field dependent. This field dependence is characterized by the critical exponent δ,[25]

$$\mu_{\rm n} \approx \chi_{\rm n} \propto H^{1/\delta - 1}. \tag{2.20}$$

The value of δ depends on the universality class the particular magnetic system belongs to. For all realistic universality classes, $\delta \approx 5$, whereas in Landau or mean-field theory, $\delta = 3$.[22] Substituting Eq. (2.20) into Eq. (2.5), we find for the thermodynamic critical field

$$H_{\rm c} \propto |t|^{2\delta/(\delta+1)} \tag{2.21}$$

This result holds for a system where the distance t from the superconducting critical point can be changed while the system remains tuned to magnetic criticality (more precisely, to the parameter values where magnetic criticality would occur in the absence of superconductivity). Generically, the dimensionless distance r from magnetic criticality will change as well if t is changed, and we will discuss such more realistic situations in Sec. IV.

For the penetration depth, the situation is more complicated. In contrast to $H_{\rm c}$, which compares the normal-state magnetic energy with the superconducting condensation energy that has nothing to do with spin magnetism, λ is entirely a property of the superconducting state, and the feedback of the superconductivity on the spin susceptibility, or the difference between $\mu_{\rm n}$ and $\mu_{\rm s}$, cannot be neglected. As a result of this feedback, the magnetic transition in the presence of superconductivity does not occur at $r = 0$, but rather at a value $r \propto -\xi_{\rm m}^0/\lambda_0$.[1] Here $\xi_{\rm m}^0$ is the magnetic correlation length at zero temperature. This suggests that the spin susceptibility at $r = 0$ will be effectively $\chi_{\rm s} \propto \lambda_0/\xi_{\rm m}^0 \gg 1$ in a mean-field approximation. More generally, one has $\mu_{\rm s} \approx \chi_{\rm s} \propto (\lambda_0/\xi_{\rm m}^0)^\gamma$, with γ another critical exponent. Using this in Eq. (2.16b) with $\mu_{\rm n}$ replaced by $\mu_{\rm s}$, we obtain

$$\lambda \propto \lambda_0^{1-\gamma/2}\left(\xi_{\rm m}^0\right)^{\gamma/2} \propto |t|^{-1/2+\gamma/4}. \tag{2.22}$$

Since $\gamma \approx 1.4 > 0$ for ferromagnetic systems,[22] this implies that the penetration depth at magnetic criticality is anomalously short. Close to the superconducting transition, the superconductor will therefore also be of type I, in agreement with a conclusion drawn from studying the ferromagnetic phase.[2]

For the critical current density, Eqs. (2.18), (2.21), and (2.22) predict

$$j_{\rm c} \propto |t|^\alpha \tag{2.23a}$$

with

$$\alpha = 2\delta/(\delta+1) + 1/2 - \gamma/4 \quad , \quad (r=0) \quad . \tag{2.23b}$$

With $\delta \approx 5$ and $\gamma \approx 1.4$ this yields $\alpha \approx 1.8$, in contrast to the Ginzburg-Landau result $\alpha = 3/2$.

These results hold at $r = 0$, and again we have assumed that t can be varied independently of r. Let us relax the former condition. From the above argument for the effective value of $\chi_{\rm s}$ at $r = 0$ it also follows that Eq. (2.22) is valid only for $|r| < \xi_{\rm m}^0/\lambda_0$. Since $\xi_{\rm m}^0$ is typically on the order of a few Å, while λ_0 is typically several hundred Å or even larger even at zero temperature, and diverges as $|t|^{-1/2}$ for $t \to 0$, this is a very small range. By contrast, Eq. (2.20) can be valid for r as large as several percent, provided H is not too small. Not too close to

T_c, where H_c goes to zero, Eq. (2.21) can thus be valid in a substantial r-range, while $\lambda = \lambda_0/\sqrt{\mu_n}$ except in an extremely small interval around $r = 0$. In that case,

$$\alpha = 2\delta/(\delta+1) + 1/2 \quad , \quad (1 \gg r \gg \xi_m^0/\lambda_0) \quad , \tag{2.23c}$$

which yields $\alpha \approx 2.17$ if $\delta \approx 5$.

Finally, at larger values of r, or sufficiently close to T_c that H_c is small enough to invalidate Eq. (2.20), we are back to the paramagnetic case, Eq. (2.19) holds, and thus $\alpha = 3/2$.

One thus faces a rather complicated situation, where the exponent α can take on values between the Ginzburg-Landau value 3/2 and a value larger than 2, Eq. (2.23c), depending on various parameters that are not easy to control or even determine experimentally. We will discuss this in more detail in Sec. IV. Before we do so, in the following section we will give a more technical and more detailed derivation of all of our results.

III. GENERALIZED GINZBURG-LANDAU THEORY

We now consider a coupled field theory that describes both superconducting and spin degrees of freedom in order to derive the above results from a more microscopic level and gain a deeper understanding of their origin. Specifically, we consider a generalization of the usual Ginzburg-Landau equations that includes the spin degrees of freedom. Far from magnetic criticality, the latter can be integrated out to yield ordinary Ginzburg-Landau theory with μ_n entering the magnetic energy density. At magnetic criticality, μ_n becomes field dependent, which changes the thermodynamic critical field. In addition, the leading term in the London equation vanishes, which leads to a generalized London equation that describes exponential decay on a length scale shorter than λ_0, in agreement with the qualitative arguments in Sec. II, and with implications for the critical current as discussed there. Unlike in the previous general discussion, in most of this section we will treat the magnetic critical behavior in a mean-field approximation.

A. LGW theory for superconducting and magnetic fluctuations

Our starting point is an action for a complex scalar field ψ describing the superconducting degrees of freedom coupled to a vector potential $\boldsymbol{A}$, and a real vector field $\boldsymbol{M}$ describing the spin degrees of freedom.[1,11] We reiterate that the qualitative behavior near the superconducting T_c does not depend on the symmetry of the order parameter, so our restriction to a scalar order parameter does not imply a loss of generality. The action reads

$$\begin{aligned} S \;=\; \int d\boldsymbol{x} \Bigg[& \frac{1}{2m}\, \left|(\boldsymbol{\nabla} - iq\boldsymbol{A}(\boldsymbol{x}))\,\psi(\boldsymbol{x})\right|^2 + \frac{t_1}{2}\,|\psi(\boldsymbol{x})|^2 + \frac{u_1}{4}\,|\psi(\boldsymbol{x})|^4 + \frac{1}{8\pi}\,\boldsymbol{B}^2(\boldsymbol{x}) \\ & + \frac{a}{2}\,(\nabla\boldsymbol{M}(\boldsymbol{x}))^2 + \frac{t_2}{2}\,\boldsymbol{M}^2(\boldsymbol{x}) + \frac{u_2}{4}\,\left(\boldsymbol{M}^2(\boldsymbol{x})\right)^2 - \boldsymbol{M}(\boldsymbol{x})\cdot\boldsymbol{B}(\boldsymbol{x}) - \frac{1}{4\pi}\,\boldsymbol{H}(\boldsymbol{x})\cdot\boldsymbol{B}(\boldsymbol{x})\Bigg]. \end{aligned} \tag{3.1}$$

Here and in the remainder of this section we use units such that Planck's constant and the speed of light are unity, $\hbar = c = 1$. The first line is the standard Landau-Ginzburg-Wilson (LGW) functional for singlet superconductors. The first three terms in the second line are a standard vector-$\boldsymbol{M}^4$ theory, with $\boldsymbol{M}(\boldsymbol{x})$ the fluctuating magnetization. $\boldsymbol{M}$ couples to the vector potential via the $\boldsymbol{M}\cdot\boldsymbol{B}$ term,[23] with $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, and the last term is necessary to relate S to the appropriate Gibbs free energy, see Eq. (2.2). Notice that ψ and $\boldsymbol{M}$ are coupled only indirectly via the vector potential $\boldsymbol{A}$. Spin-flip scattering of electrons by the magnetic moments does give rise to a direct coupling of the form $\boldsymbol{M}^2|\psi|^2$,[2] but these terms are not important for our purposes.

Minimizing this action with respect to ψ^*, $\boldsymbol{A}$, and $\boldsymbol{M}$ yields the following saddle-point equations,

$$t_1\,\psi(\boldsymbol{x}) + u_1\,|\psi(\boldsymbol{x})|^2\psi(\boldsymbol{x}) - \frac{1}{m}\,(\boldsymbol{\nabla} - iq\boldsymbol{A}(\boldsymbol{x}))^2\,\psi(\boldsymbol{x}) \;=\; 0, \tag{3.2a}$$

$$-i\,\frac{q}{2m}\,[\psi^*(\boldsymbol{x})\boldsymbol{\nabla}\psi(\boldsymbol{x}) - \psi(\boldsymbol{x})\boldsymbol{\nabla}\psi^*(\boldsymbol{x})] - \frac{q^2}{m}\,|\psi(\boldsymbol{x})|^2\boldsymbol{A}(\boldsymbol{x}) \;=\; \frac{1}{4\pi}\,\boldsymbol{\nabla}\times[\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{H}(\boldsymbol{x}) - 4\pi\boldsymbol{M}(\boldsymbol{x})]\,, \tag{3.2b}$$

$$t_2\,\boldsymbol{M}(\boldsymbol{x}) - a\,\boldsymbol{\nabla}^2\boldsymbol{M}(\boldsymbol{x}) + u_2\,\boldsymbol{M}^2(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x}) \;=\; \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x}). \tag{3.2c}$$

If we drop Eq. (3.2c) and put $\boldsymbol{M} = 0$ in Eq. (3.2b) (this corresponds to dropping $\boldsymbol{M}$ from the action) we recover the usual Ginzburg-Landau equations.[20] A non-superconducting solution of the full equations is $\psi = 0$, $\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}$, and $\boldsymbol{M}$ determined by the magnetic

equation of state

$$\left(r - a\boldsymbol{\nabla}^2\right)\boldsymbol{M}(\boldsymbol{x}) + u_2\,\boldsymbol{M}^2(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{H}(\boldsymbol{x}), \tag{3.3}$$

where $r = t_2 - 4\pi$. For a small constant external field $\boldsymbol{H}$ a solution of Eq. (3.3) is $\boldsymbol{M} = \chi_{\rm n}\,\boldsymbol{H}$, with

$$\chi_{\rm n} = 1/r \tag{3.4}$$

the normal-state magnetic susceptibility (see Ref. 23). At this point it is the bare susceptibility, but it is clear that by renormalizing the spin part of the action before constructing the saddle-point solution one can make it the physical susceptibility.

B. Effective theory for paramagnetic superconductors

Now consider the full Eqs. (3.2). For a small and slowly varying $\boldsymbol{M}(\boldsymbol{x})$ we have from Eq. (3.2c)

$$\boldsymbol{M}(\boldsymbol{x}) = \left(\chi_{\rm n}^{-1} + 4\pi\right)^{-1}\boldsymbol{B}(\boldsymbol{x}). \tag{3.5}$$

Substituting this into Eq. (3.2b) we obtain

$$\boldsymbol{j}_{\rm sc}(\boldsymbol{x}) = \frac{1}{4\pi\mu_{\rm n}}\,\boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{x}) - \frac{1}{4\pi}\,\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{x}), \tag{3.6a}$$

where

$$\begin{aligned}\boldsymbol{j}_{\rm sc}(\boldsymbol{x}) &= -i\,\frac{q}{2m}\,\left[\psi^*(\boldsymbol{x})\boldsymbol{\nabla}\psi(\boldsymbol{x}) - \psi(\boldsymbol{x})\boldsymbol{\nabla}\psi^*(\boldsymbol{x})\right]\\ &\quad -\frac{q^2}{m}\,|\psi(\boldsymbol{x})|^2\boldsymbol{A}(\boldsymbol{x}).\end{aligned} \tag{3.6b}$$

Together with Eq. (3.2a), these are the equations of motion for an effective action[19]

$$\begin{aligned}S_{\rm eff} &= \int d\boldsymbol{x}\,\Big[\frac{1}{2m}\,\left|\left(\boldsymbol{\nabla} - iq\boldsymbol{A}(\boldsymbol{x})\right)\psi(\boldsymbol{x})\right|^2 + \frac{t}{2}\,|\psi(\boldsymbol{x})|^2\\ &\quad +\frac{u}{4}\,|\psi(\boldsymbol{x})|^4 + \frac{1}{8\pi\mu_{\rm n}}\,\boldsymbol{B}^2(\boldsymbol{x}) - \frac{1}{4\pi}\,\boldsymbol{H}(\boldsymbol{x})\cdot\boldsymbol{B}(\boldsymbol{x})\Big],\end{aligned} \tag{3.7}$$

where we have dropped the now-superfluous subscript on the Landau parameters t and u. The same result is of course obtained by starting with Eq. (3.1) and integrating out $\boldsymbol{M}$ in a Gaussian approximation.

The quantity $\boldsymbol{j}_{\rm sc}$ in Eqs. (3.6) is indeed the supercurrent, as can be seen by comparing Eq. (3.6a) with Eq. (2.9a). It does not explicitly depend on $\mu_{\rm n}$, see Eq. (3.6b), and this is important for the flux quantum to be independent of μ_n. The magnetic energy $\boldsymbol{B}^2/8\pi\mu_{\rm n}$, which does explicitly depend on $\mu_{\rm n}$, does not appreciably contribute to the free energy of a thin film or wire sample, and the standard determination of the critical current, Ref. 20, thus leads to the usual Ginzburg-Landau result with no correction due to $\mu_{\rm n} \neq 1$. This corroborates the educated guess in Sec. II A 3.

For all other quantities, the usual analysis of Ginzburg-Landau theory now applies.[20] One characteristic length scale is given by the square root of the ratio of the coefficients of the gradient-squared term and the ψ^2 term in Eq. (3.7). This is the superconducting coherence length $\xi = \sqrt{1/m|t|}$. Another one is given by the square root of the ratio of the coefficients of the terms quadratic in $\boldsymbol{A}$. For a constant ψ, this is the London penetration depth

$$\lambda = \sqrt{m/4\pi q^2\psi^2\mu_{\rm n}} \equiv \lambda_0/\sqrt{\mu_{\rm n}}. \tag{3.8}$$

This is identical with Eq. (2.16b), which had been deduced on elementary phenomenological grounds.

For the Ginzburg-Landau parameter $\kappa = \lambda/\xi$ we now have $\kappa = \kappa_0/\sqrt{\mu_{\rm n}}$, with κ_0 the value of the parameter for $\mu_{\rm n} = 1$. This implies that the superconductor is of type I or type II, respectively, for $\kappa_0 < \sqrt{\mu_{\rm n}/2}$ or $\kappa_0 > \sqrt{\mu_{\rm n}/2}$. While one can show this by an explicit analysis of the effective action, a fast way to relate the theory for arbitrary values of $\mu_{\rm n}$ to the one for $\mu_{\rm n} = 1$ is to rewrite the action in terms of dimensionless quantities.[24] In conventional Ginzburg-Landau theory, this is done by introducing

$$\boldsymbol{x} = \lambda_0\,\hat{\boldsymbol{x}}\ ,\ \psi(\boldsymbol{x}) = \psi_0\,\hat{\psi}(\hat{\boldsymbol{x}})\ ,\ \boldsymbol{A}(\boldsymbol{x}) = \sqrt{2}H_{\rm c}^0\lambda_0\hat{\boldsymbol{A}}(\hat{\boldsymbol{x}}),. \tag{3.9}$$

Here $\psi_0 = \sqrt{-t/u}$ is the superconducting order parameter scale. In terms of these quantities, the effective action reads[24]

$$\begin{aligned}S_{\rm eff} &= \frac{(H_{\rm c}^0)^2\lambda_0^3}{4\pi}\int d\hat{\boldsymbol{x}}\,\Big[\Big|\Big(\frac{1}{\kappa_0}\,\hat{\boldsymbol{\nabla}} - i\hat{\boldsymbol{A}}(\hat{\boldsymbol{x}})\Big)\,\hat{\psi}(\hat{\boldsymbol{x}})\Big|^2\\ &\quad -|\hat{\psi}(\hat{\boldsymbol{x}})|^2 + \frac{1}{2}\,|\hat{\psi}(\hat{\boldsymbol{x}})|^4 + \frac{1}{\mu_{\rm n}}\,\Big(\hat{\boldsymbol{\nabla}}\times\hat{\boldsymbol{A}}(\hat{\boldsymbol{x}})\Big)^2\\ &\quad -2\hat{\boldsymbol{H}}(\hat{\boldsymbol{x}})\cdot\Big(\hat{\boldsymbol{\nabla}}\times\hat{\boldsymbol{A}}(\boldsymbol{x})\Big)\Big],\end{aligned} \tag{3.10}$$

A simple further rescaling procedure shows that $S_{\rm eff}$ depends only on a single dimensionless parameter, rather than the two parameters κ_0 and $\mu_{\rm n}$. Define

$$\hat{\boldsymbol{x}} = \tilde{\boldsymbol{x}}/\kappa_0\ ,\ \hat{\boldsymbol{A}}(\tilde{\boldsymbol{x}}/\kappa_0) = \tilde{\boldsymbol{A}}(\tilde{\boldsymbol{x}})\ ,\ \hat{\psi}(\tilde{\boldsymbol{x}}/\kappa_0) = \tilde{\psi}(\tilde{\boldsymbol{x}}). \tag{3.11}$$

Then

$$\begin{aligned}S_{\rm eff} &= \frac{(H_{\rm c}^0)^2\xi^3}{4\pi}\int d\tilde{\boldsymbol{x}}\,\Big[\Big|\Big(\tilde{\boldsymbol{\nabla}} - i\tilde{\boldsymbol{A}}(\tilde{\boldsymbol{x}})\Big)\,\tilde{\psi}(\tilde{\boldsymbol{x}})\Big|^2\\ &\quad -|\tilde{\psi}(\tilde{\boldsymbol{x}})|^2 + \frac{1}{2}\,|\tilde{\psi}(\tilde{\boldsymbol{x}})|^4 + \frac{\kappa_0^2}{\mu_{\rm n}}\,\Big(\tilde{\boldsymbol{\nabla}}\times\tilde{\boldsymbol{A}}(\tilde{\boldsymbol{x}})\Big)^2\\ &\quad -2\frac{\kappa_0}{\sqrt{\mu_{\rm n}}}\,\Big(\sqrt{\mu_{\rm n}}\tilde{\boldsymbol{H}}(\tilde{\boldsymbol{x}})\Big)\cdot\Big(\tilde{\boldsymbol{\nabla}}\times\tilde{\boldsymbol{A}}(\tilde{\boldsymbol{x}})\Big)\Big].\end{aligned} \tag{3.12}$$

This shows that the theory with an arbitrary $\mu_{\rm n}$ maps onto ordinary Ginzburg-Landau theory with the replacements

$$\kappa_0 \to \kappa_0/\sqrt{\mu_{\rm n}} \equiv \kappa \quad , \quad \boldsymbol{H} \to \sqrt{\mu_{\rm n}}\,\boldsymbol{H}. \tag{3.13}$$

$\kappa_0 = \sqrt{\mu_{\rm n}/2}$ thus indeed marks the demarcation between type I and type II superconductors, and the critical fields can be immediately obtained from the usual results at $\mu_{\rm n} = 1$.[20] For the thermodynamic critical field $H_{\rm c}$, the upper critical field $H_{\rm c2}$, the lower critical field $H_{\rm c1}$, and the surface critical field $H_{\rm c3}$ we obtain

$$\begin{aligned} H_{\rm c} &= H_{\rm c}^0/\sqrt{\mu_{\rm n}}, && (3.14a)\\ H_{\rm c2} &= H_{\rm c2}^0/\mu_{\rm n} = \sqrt{2}\kappa_0 H_{\rm c}^0/\mu_{\rm n}, && (3.14b)\\ H_{\rm c1} &= H_{\rm c1}^0 \frac{g(\kappa_0/\sqrt{\mu_{\rm n}})}{g(\kappa_0)} = \frac{H_{\rm c}^0}{\sqrt{2}\kappa_0}\, g(\kappa_0/\sqrt{\mu_{\rm n}}). && (3.14c)\\ H_{\rm c3} &= 1.695\, H_{\rm c2}. && (3.14d) \end{aligned}$$

where the universal function g has the limiting behavior

$$g(x) = \begin{cases} \ln x + 0.08 + O(1/x) & \text{for} \quad x \gg 1/\sqrt{2}, \\ 1 & \text{for} \quad x = 1/\sqrt{2}. \end{cases} \tag{3.14e}$$

If one neglects the weak dependence of g on its argument, $H_{\rm c1}$ is approximately independent of $\mu_{\rm n}$.

C. Superconductors at magnetic criticality

As one approaches a ferromagnetic instability, $\mu_{\rm n}$ keeps increasing and can no longer be treated as a constant. There are two effects that become important for our purposes. First, in a normal metal $\mu_{\rm n}$ becomes strongly field or induction dependent. At $r = 0$ this dependence is nonanalytic and described by the critical exponent δ. Second, as r becomes on the order of $\xi_{\rm m}^0/\lambda$ (see Sec. II B) in a superconducting phase, the difference between $\mu_{\rm n}$ and $\mu_{\rm s}$ can no longer be neglected. Related to this, the gradient squared term in Eq. (3.2c) must be taken into account. We now consider these effects, starting with the nonanalytic field dependence in the normal state.

1. *Thermodynamic critical field*

At magnetic criticality in the normal state, $r = 0$, one has[25]

$$\chi_{\rm n}(r=0, H) = \chi_0 \left(H/\tilde{H}_0\right)^{1/\delta - 1}. \tag{3.15}$$

Here χ_0 is a microscopic susceptibility, and $\tilde{H}_0$ is a microscopic field scale. M and, for small values of H, B are therefore proportional to $H^{1/\delta}$, or $H \propto B^\delta$. For small B, the number $\mu_{\rm n}$ should thus be replaced by a function of B with the following leading B-dependence,

$$\mu_{\rm n} \to (H_0/B)^{\delta-1}, \tag{3.16}$$

with $H_0 = (4\pi\chi_0)^{\delta/(\delta-1)}\tilde{H}_0$. The magnetic energy cost of the flux expulsion that results from the formation of a Meissner phase (which equals minus the normal-state magnetic energy) is now obtained by using Eq. (3.16) in Eq. (3.7). It is $E_{\rm m}/V = HB/4\pi - B^{\delta+1}/8\pi H_0^{\delta-1} = H_0^{1-1/\delta} H^{1+1/\delta}/8\pi$. The condensation energy is still given by $E_{\rm cond}/V = t^2/4u$, which yields

$$H_{\rm c} = \left(\frac{2\pi}{u}\right)^{\delta/(\delta+1)} \frac{1}{H_0^{(\delta-1)/(\delta+1)}}\, |t|^{2\delta/(\delta+1)}. \tag{3.17}$$

The thermodynamical critical field is thus weaker than in the paramagnetic case, and the t-dependence is consistent with Eq. (2.21). By comparing with Eq. (2.5), we see that with respect to the thermodynamical critical field, $\mu_{\rm n}$ effectively scales like $\mu_{\rm n} \sim 1/|t|^{2(\delta-1)/(\delta+1)}$ at magnetic criticality.

Equations (3.15) through (3.17) hold also for small but nonzero values of r as long as one is in the field scaling regime, i.e, as long as H in appropriate units is large compared to r to an appropriate power. We will discuss this in more detail in Sec. IV. At this point we only mention that, since $H_{\rm c}$ vanishes as $|t| \to 0$, sufficiently close to $T_{\rm c}$ one will lose the field scaling for any nonzero value of r, and $H_{\rm c}$ will be given by Eq. (3.14a).

2. *Generalized London equation*

The ordinary London equation is obtained from Eq. (3.2b) by dropping $\boldsymbol{M}(\boldsymbol{x})$ and treating $\psi(\boldsymbol{x}) \equiv \psi$ as a constant (London approximation). With $\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{x}) = 0$ this leads to

$$-\lambda_0^{-2}\, \boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{x}). \tag{3.18}$$

Now take $\boldsymbol{M}$ into account. Using Eq. (3.2c) in Eq. (3.2b), we can eliminate $\boldsymbol{B}$ and derive an equation for $\boldsymbol{M}$. Once $\boldsymbol{M}$ is known, $\boldsymbol{B}$ follows from Eq. (3.2c). Within the London approximation one finds

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{x}) \;=\; & -(\lambda_0^2/\mu_{\rm n})\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{M}(\boldsymbol{x}) + \left(\tilde{\xi}_{\rm m}^0\right)^2 \boldsymbol{\nabla}^2 \boldsymbol{M}(\boldsymbol{x}) + \left(\tilde{\xi}_{\rm m}^0\right)^2 \lambda_0^2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\nabla}^2 \boldsymbol{M}(\boldsymbol{x}) - \tilde{u}\, \boldsymbol{M}^2(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x}) \\ & -\tilde{u}\lambda_0^2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{M}^2(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x}). \end{aligned} \tag{3.19}$$

Here $\mu_{\rm n} = (4\pi{+}r)/r$ as in Sec. III B, $\tilde{\xi}_{\rm m}^0 = \xi_{\rm m}^0/\sqrt{4\pi + r} \equiv \sqrt{a/(4\pi + r)}$, and $\tilde{u} = u/(4\pi + r)$.

As long as $\mu_{\rm n} \approx 1$, the first term on the right-hand side of Eq. (3.19) leads to a variation of $\boldsymbol{M}$ on a length scale

$\lambda = \lambda_0/\sqrt{\mu_{\rm n}}$. The second term is a small correction to the first one since $\xi_{\rm m}^0 \ll \lambda_0$. So is the third term, which is of order $\left(\tilde{\xi}_{\rm m}^0\right)^2 \boldsymbol{\nabla}^2 \sim \left(\tilde{\xi}_{\rm m}^0\right)^2/\lambda_0^2 \ll 1$ relative to the first one. The linearized version of Eq. (3.19) thus reduces to the ordinary London equation, Eq. (3.18), with $\lambda_0 \to \lambda$. However, for $r = 0$ the first term vanishes. This makes the second term the leading one, and the third term, which is of order $\lambda_0^2\boldsymbol{\nabla}^2$ compared to the second one, cannot be neglected either. The linearized equation thus reads

$$\boldsymbol{M}(\boldsymbol{x}) = \left(\tilde{\xi}_{\rm m}^0\right)^2 \boldsymbol{\nabla}^2 \left[1 + \lambda_0^2 \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\right] \boldsymbol{M}(\boldsymbol{x}). \quad (3.20)$$

With the same interface geometry as in Sec. II A 2 this takes the form

$$M(x) = \left(\tilde{\xi}_{\rm m}^0\right)^2 M''(x) - \left(\tilde{\xi}_{\rm m}^0\right)^2 \lambda_0^2 M^{(iv)}(x). \quad (3.21)$$

This linear quartic ODE is solved by an exponential ansatz, $M(x) = M_0 e^{-\rho x}$. The real solution that falls off for $x \to \infty$ shows damped oscillatory behavior. From Eq. (3.2c) we see that $B(x)$ shows the same behavior as $M(x)$, up to corrections of $O(\tilde{\xi}_{\rm m}^0/\lambda_0)$. With the boundary condition $B(x=0) = \mu_{\rm n}H$ we finally obtain

$$B(x) = \mu_{\rm n} H e^{-x/\sqrt{2\tilde{\xi}_{\rm m}^0\lambda_0}} \cos\left(x/\sqrt{2\tilde{\xi}_{\rm m}^0\lambda_0}\right) . \quad (3.22)$$

This is the solution of the linearized version of Eq. (3.19) at $r = 0$. In addition to leaving out the terms of $O(M^3)$, we have also ignored the fact that the permeability, whether $\mu_{\rm n}$ or $\mu_{\rm s}$, does depend on B or M at magnetic criticality. In a mean-field approximation, $\mu_{\rm n} \propto 1/B^2$ at $r = 0$, see Eq. (3.16), which also leads to terms of $O(M^3)$ in the nonlinear equation. Depending on the ratio of the external field to H_0, these terms may or may not be important for the initial decay of M or B near the normal metal-to-superconductor boundary. However, once M or B has decayed sufficiently, these terms become subleading compared to the linear ones in Eq. (3.21), and the asymptotic behavior as $B \to 0$ is always given by Eq. (3.22).

In order to make contact with the discussion in Sec. II B for small but nonzero values of r, let us consider the linearized Eq. (3.19) while keeping the first term. Instead of Eq. (3.21) we then have

$$M(x) = \left(\lambda_0^2/\mu_{\rm n} + \left(\tilde{\xi}_{\rm m}^0\right)^2\right) M''(x) - \left(\tilde{\xi}_{\rm m}^0\right)^2 \lambda_0^2 M^{(iv)}(x). \quad (3.23)$$

This is solved by

$$M(x) = M_0\, e^{\rho x}, \quad (3.24a)$$

with

$$\rho^2 = \frac{1}{2\lambda_0^2\left(\tilde{\xi}_{\rm m}^0\right)^2}\left[\lambda_0^2/\mu_{\rm n} + \left(\tilde{\xi}_{\rm m}^0\right)^2 - \sqrt{\left(\lambda_0^2/\mu_{\rm n} + \left(\tilde{\xi}_{\rm m}^0\right)^2\right)^2 - 4\lambda_0^2\left(\tilde{\xi}_{\rm m}^0\right)^2}\right]. \quad (3.24b)$$

Here we have chosen the solution for ρ^2 that yields $\rho^2 \to 1/\lambda_0^2$ for $r \to \infty$. Equation (3.24b) still provides two solutions for ρ, and the physical solution for M is determined by the requirement that M be real.

A discussion of Eq. (3.24b) shows that ρ^2 becomes purely real and negative at $r = r_{\rm s} = -4\sqrt{\pi}\tilde{\xi}_{\rm m}^0/\lambda_0 + O\left(\left(\tilde{\xi}_{\rm m}^0\right)^2/\lambda_0^2\right)$. This is in agreement with the results of Blount and Varma,[1] who showed that spiral magnetic order coexisting with the superconductivity occurs at this point. For $|r| \ll \tilde{\xi}_{\rm m}^0/\lambda_0$ one has $\rho^2 \approx -i/\tilde{\xi}_{\rm m}^0\lambda_0$, which leads to Eq. (3.22). For $r \gg \tilde{\xi}_{\rm m}^0/\lambda_0$ one finds $\rho^2 \approx \mu_{\rm n}/\lambda_0^2$, which leads to

$$B(x) = \mu_{\rm n} H e^{-x\sqrt{\mu_{\rm n}}/\lambda_0}, \quad (3.25)$$

in agreement with Eq. (3.8).

3. Penetration depth, and critical current

Equation (3.22) shows that the effective penetration depth at magnetic criticality is

$$\lambda = \sqrt{2\tilde{\xi}_{\rm m}^0\lambda_0} \quad , \quad (|r| \ll \tilde{\xi}_{\rm m}^0/\lambda_0), \quad (3.26)$$

in agreement with the conclusions of Ref. 2 drawn from studying the ferromagnetic phase, and with Eq. (2.22) with $\gamma = 1$. The latter approximation results from the fact that our saddle-point equations of motion describe the magnetic equation of state in a mean-field approximation. The discussion of Eq. (3.24b) shows that this result is valid for $|r| \ll \tilde{\xi}_{\rm m}^0/\lambda_0$. By comparing with Eq. (2.16b) or (3.8), we see that with respect to the penetration depth, $\mu_{\rm n}$ at magnetic criticality scales like $\mu_{\rm n} \sim 1/\sqrt{|t|}$ in mean-field approximation, or $\mu_{\rm n} \sim 1/|t|^{\gamma/2}$ in general. The fact that $1/\sqrt{\mu_{\rm n}}$ in Eqs. (2.5) and (2.16b), respectively, must be interpreted differently for $\mu_{\rm n} \to \infty$ is a consequence of the influence of the superconductivity on the spin response.

For $r \gg \tilde{\xi}_{\rm m}^0/\lambda_0$ we have, from Eq. (3.25)

$$\lambda = \lambda_0/\sqrt{\mu_{\rm n}}, \quad (3.27)$$

in agreement with Eq. (3.8).

The expression for the critical current given by Eq. (2.18) is general within the London approximation. We have now given a derivation of the behavior of the thermodynamical critical field and the penetration depth given on phenomenological grounds in Eqs. (2.21) and (2.22), respectively. The behavior of the critical current at or near magnetic criticality is thus given by Eqs. (2.23).

4. Critical field H_{c2}

The critical exponent γ is positive ($\gamma \approx 1.4$ for typical ferromagnetic universality classes in three dimensions[22]).

The result for λ, Eq. (2.22) or (3.23) in mean-field approximation, of the previous subsection therefore means that λ diverges more slowly for $|t| \to 0$ than the superconducting coherence length $\xi \propto 1/\sqrt{|t|}$. Consequently, superconductors at magnetic criticality ($|r| \ll \tilde{\xi}_{\rm m}^0/\lambda_0$) are necessarily of type I.[2]

This observation notwithstanding, the critical field $H_{\rm c2}$, which in a type-II superconductor signalizes the boundary of the vortex phase, still has a physical meaning: It is the minimum field to which the normal metal can be 'supercooled' before it discontinuously develops a nonzero superconducting order parameter.[20] It is thus still of interest to determine $H_{\rm c2}$. Furthermore, the behavior will be necessarily of type I only for $|r|$ in an extremely narrow region. Outside of this region, Eq. (3.27) holds, and for a sufficiently large value of $\kappa_0 = \lambda_0/\xi$ the superconductor will still be of type II. The determination of $H_{\rm c2}$ is done by linearizing the Ginzburg-Landau equation, Eq. (3.2a), in ψ. It then turns into a Schrödinger equation for a particle in a vector potential $\boldsymbol{A}$, with $-t_1/2 \equiv -t/2$ playing the role of the energy eigenvalue. By means of standard arguments[20] this leads to a critical value of the magnetic induction $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{H}$ given by $B_{\rm c2} \equiv H_{\rm c2}^0 = -tm/q$. In a paramagnetic superconductor, this leads to

$$H_{\rm c2} = H_{\rm c2}^0/\mu_{\rm n} \quad (\mu_{\rm n} = \text{const.}), \tag{3.28}$$

which is the same as Eq. (3.14b). At magnetic criticality, we have, cf. Eq. (3.16),

$$H_{\rm c2} = B_{\rm c2}^{\delta}/H_0^{(\delta-1)} \propto |t|^{\delta}. \tag{3.29}$$

Notice that, in this context, $\mu_{\rm n}$ scales as $\mu_{\rm n} \sim 1/|t|^{\delta-1}$, whereas it scales as $\mu_{\rm n} \sim 1/|t|$ if the relevant field scale is $H_{\rm c}$. Since $H_{\rm c2}$ vanishes much faster than $H_{\rm c}$, Eq. (3.17), the field scaling region will be restricted to larger values of $|t|$, and $H_{\rm c2}$ will be given by Eq. (3.28) in a substantial range of t-values. Will come back to this in Sec. IV.

IV. DISCUSSION AND CONCLUSION

To summarize, we have determined the electrodynamic properties of superconductors close to a ferromagnetic instability, i.e., materials that, in the absence of superconductivity, would be paramagnetic with large ferromagnetic fluctuations. This work complements previous studies of the coexistence of superconductivity with ferromagnetic order.[1,2] We have treated the superconductivity in mean-field (Ginzburg-Landau) approximation. In addition, we have employed the London approximation, treating the superconducting order parameter as a constant. The ferromagnetic critical point we have treated explicitly in a mean-field approximation, and we have used scaling arguments to consider the consequences of the exact magnetic critical behavior for the superconductivity. We have found that the thermodynamical critical field $H_{\rm c}$ decreases due to the ferromagnetic fluctuations, as one would expect, and depends on the magnetic critical exponent δ, see Eqs. (3.17) and (2.21). However, the London penetration depth also decreases, which is intuitively less obvious. At magnetic criticality the behavior of the magnetic induction at a vacuum-to-superconductor (or normal metal-to-superconductor) interface is still characterized by exponential decay, but the characteristic length scale λ is different from the usual London penetration depth λ_0. Within a mean-field description of the magnetic criticality it is the geometric mean of the zero-temperature magnetic correlation length and λ_0, see Eqs. (3.22) and (3.26); more generally, it depends on the magnetic critical exponent γ, see Eq. (2.22). However, this behavior of the penetration depth is valid only within an extremely small region of width $\tilde{\xi}_{\rm m}^0/\lambda_0$ around magnetic criticality. Outside of this region, but still within the ferromagnetic critical region, the temperature dependence of the penetration depth is the same as in Ginzburg-Landau theory, see Eq. (3.27). For the critical current $j_{\rm c} \propto H_{\rm c}/\lambda$ this implies a dependence on the reduced temperature given by $|t|^{\alpha}$, where the exponent α depends on both δ and γ, or on δ only, depeding on the value of r, see Eqs. (2.23). With exponent values appropriate for the usual ferromagnetic universality classes, $\alpha \approx 1.8$ extremely close to magnetic criticality, and $\alpha \approx 2.15$ somewhat farther away.

Let us now discuss these results in some more detail, and relate them to the experimental observations reported in Ref. 16.

For the temperature dependencies of various observables at magnetic criticality we have assumed that the system stays tuned to magnetic criticality while the temperature is varied. Let us discuss to what extent this assumption is realistic. Consider a phase diagram in a plane spanned by the temperature and some non-thermal control parameter x, e.g., the hole doping concentration in the case of $MgCNi_3$,[15] and consider the following two qualitatively different possibilities. Figure 2 shows a situation where the magnetic phase separation line does not cross the line $x = 0$. The stoichiometric compound thus does not enter a magnetic phase upon cooling, although the system is close to a magnetic transition for all temperatures below the superconducting $T_{\rm c}$. This scenario is believed to apply to $MgCNi_3$. Figure 3 shows a situation where the magnetic phase separation line does cross the line $x = 0$, so that the stoichiometric compound enters a phase where superconductivity and magnetism coexist at some temperature below $T_{\rm c}$. This is the situation that was discussed in Refs. 1 and 2 and observed in $ErRh_4B_4$ and $HoMo_6S_8$.[4,5] The magnetic transition is to a phase with spiral magnetic order at a temperature $T_{\rm s}$ slightly below the temperature $T_{\rm m}^0$ where ferromagnetism would occur in the absence of superconductivity.[1]

We now can see what is required to keep r constant while varying t, namely, a situation as shown in Fig. 2 with the dashed line essentially parallel to the T-axis. r is then given by the dimensionless distance between the two lines. In order for the penetration depth to display the

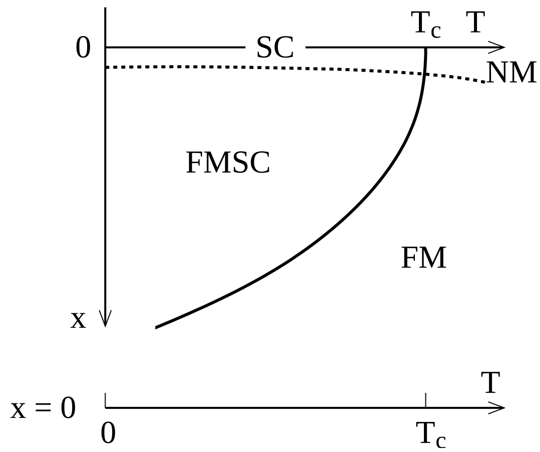

FIG. 2: Schematic phase diagram showing a normal metal (NM), a ferromagnet (FM), a superconductor (SC), and a ferromagnetic superconductor (FMSC) in a temperature (T) - control parameter (x plane. The solid line denotes the superconducting transition, the dashed line, the magnetic one. Along $x = 0$ there is only one phase transition at the superconducting $T_{\rm c}$. See the text for additional explanation.

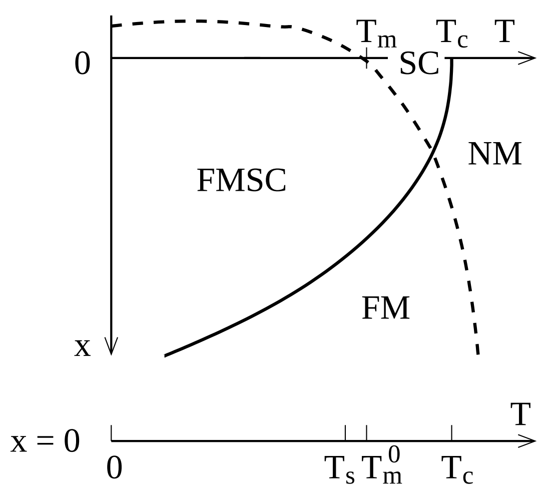

FIG. 3: Same as Fig. 2, but with a magnetic transition for $x = 0$ at a temperature $T_{\rm m} < T_{\rm c}$. On the $x = 0$ axis it is shown that $T_{\rm m}$ splits into the bare magnetic transition temperature $T_{\rm m}^0$ and the physical transition temperature $T_{\rm s}$ to a state with spiral magnetic order, Ref.1. See the text for additional explanation.

non-Ginzburg-Landau behavior described by Eq. (3.26) or, more generally, Eq. (2.22), the two lines would have to be extremely close, in order to keep r smaller than $\tilde{\xi}_{\rm m}^0/\lambda_0$, see Eq. (3.26). This would result in a temperature dependence of the critical current given by Eqs. (2.23a, 2.23b). While this is possible, it is a very non-generic situation, and it would result in a very large magnetic susceptibility of the normal metal just above the superconducting transition temperature.

A situation that is still very non-generic, but requires somewhat less fine-tuning, is one where the dashed line is still essentially parallel to the T-axis, but in a somewhat larger r-range, say, with r on the order of a few percent. In this case the penetration depth will show the usual $1/\vert t\vert^{1/2}$ temperature dependence, see Eq. (3.27). The temperature dependence of the thermodynamic critical field will be more complicated in this case. The generalization of Eq. (2.20) to nonzero values of r is

$$\chi_{\rm n} = r^{-\gamma}\, f_{\chi}(H/r^{\beta\delta}), \tag{4.1}$$

with $\gamma = \beta(\delta - 1)$, β, and δ the usual critical exponents for the magnetic transition. In order for Eq. (2.20) to hold, the H must be large compared to $r^{\beta\delta}$ in suitable units. The latter are not determined by any universal arguments, but an analysis of the critical equation of state for both the high-temperature ferromagnet Ni ($T_{\rm m} \approx 630\,{\rm K}$)[26] and the low-temperature ferromagnet $CrBr_3$ ($T_{\rm m} \approx 33\,{\rm K}$)[27] shows that in either case the relevant energy or field scale (we use units such that $k_{\rm B} = \mu_{\rm B} = 1$) is given by $T_{\rm m}$, which is plausible. The crossover between the field scaling that leads to Eq. (2.21) and the static scaling that leads to Eq. (2.5) thus occurs at a crossover field

$$H_{\times} \approx T_{\rm m}^0\, r^{\beta\delta}. \tag{4.2}$$

$\beta\delta \approx 5/3$ for ferromagnetic phase transitions, and with $T_{\rm m} \approx 10\,{\rm K}$ and $r \approx 0.1$, one finds $H_{\times} \approx 0.02\,T_{\rm m}^0$. For $MgCNi_3$ in the vicinity of $T_{\rm c}$, this leads to $H_{\times} \approx 0.2\,{\rm T}$. With H_{c2} at *zero temperature* on the order of 14 T and $\kappa \approx 40$,[28] one expects $H_{\rm c}(T = 0) = H_{c2}/\sqrt{2}\kappa \approx 0.25\,{\rm T}$. Since $H_{\rm c}$ vanishes at $T_{\rm c}$, this means that $H_{\rm c}$ will be given by Eq. (2.21) sufficiently far away from $T_{\rm c}$, but cross over to $H_{\rm c} \propto \vert t\vert$ near $T_{\rm c}$. Consequently, the critical current exponent α will be given by Eq. (2.23c) at some distance from $T_{\rm c}$, and cross over to the Ginzburg-Landau result $\alpha = 3/2$ as $\vert t\vert \to 0$. In the experiment of Ref. 16, no such crossover was observed down to $\vert t\vert \approx 0.01$.

At least within the London approximation, our results confirm the conclusion of Ref. 2 that superconductors near a ferromagnetic instability are necessarily of type I. However, we have also shown that this conclusion is inevitable only within an extremely small region around the (bare) magnetic critical point. The fact that $MgCNi_3$ is observed to be of type II[28] is therefore not necessarily in contradiction to the notion that this material is almost ferromagnetic. However, Eq. (3.29) predicts a strong deviation from Ginzburg-Landau behavior for the upper critical field $H_{\rm c2}$. Since $H_{\rm c2}$ goes to zero rapidly as $\vert t\vert \to 0$, this behavior will show only at substantial values of $\vert t\vert$ even if r is very small. No anomalous behavior was observed for $\vert t\vert$ up to 0.5.[28] This is reconcilable with close proximity to a magnetic instability only if r is very small close to $T_{\rm c}$, and grows with decreasing temperature, in which case $H_{\rm c2}$ might never show the magnetic critical behavior. A signature of this situation would be a large magnetic susceptibility in the normal state just above $T_{\rm c}$.

The conclusion from this discussion with respect to the experimental observations in Ref. 16 is as follows. While

it is possible that proximity to a ferromagnetic instability is the cause of the observed anomalous behavior of the critical current, such an explanation requires fine tuning of the phase diagram, and would have to be accompanied by a very large enhancement of the spin susceptibility in the normal phase just above T_c. Explaining the lack of an anomaly in the temperature dependence of H_{c2} probably requires that the material is closer to the magnetic instability near T_c than at $T = 0$ (i.e., the dashed line in Fig. 2 comes closer to the T-axis with increasing T). A direct measurement of the spin susceptibility in the normal phase would be of great interest in this context.

Finally, we discuss our predictions for the case of a superconductor that does undergo a transition to a magnetic state below T_c, i.e., the situation represented by Fig. 3. In the (very small) temperature interval of width $2|T_m^0 - T_s|$ around T_m^0, both the thermodynamic critical field H_c and the penetration depth λ will show an anomalous temperature dependence, and the critical current exponent will be given by Eq. (2.23b). Outside of this region, but not too close to T_c, H_c will be anomalous, but λ will be conventional, and the critical current exponent will be given by Eq. (2.23c). Upon approaching T_c, H_c will fall below the crossover field given by Eq. (4.2), and its temperature dependence will cross over to the usual linear Ginzburg-Landau behavior. The critical current exponent close to T_c will thus be the conventional $\alpha = 3/2$. The location of this crossover depends on the critical field scale, and will thus be material dependent. Critical current measurements in the materials like $ErRh_4B_4$, or $HoMo_6S_8$, which are believed to fall into this class, would be very interesting.

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